**EMS**

**European Mathematical Society**

**Committee on Mathematics Education**

**Reference levels in**
**School Mathematics Education in Europe for age 16**

**Niveaux de référence pour l'enseignement des**

**mathématiques en Europe à l'âge de 16 ans**

# REFERENCE QUESTIONS
## (''DREAM QUESTIONS'')
## Part 1

*Antoine Bodin and Vinicio Villani*
**August, 2001**

# 1 - Introduction

The main report concerning the reference level project (international report - May 2001) intends mainly to provide references concerning curriculum issues (intended and implemented mathematics curricula in the countries concerned by the study).

Students' achievement issues have been only slightly touched by the main report. Concerning these issues, at least four questions may be raised :

> 1 - What 16 year old students really know, in mathematics, across Europe ?

> 2 - How are they assessed ?

> 3 - What is worth for them to know ?

> 4 - Is it possible to design any assessment scheme that could be used across countries without regard to the many differences that have been acknowledged in the main report ?

This paper summarises our reflections about these questions and, at the same time, introduces a bunch of possible reference questions (reference questions - part 2).

# 2 - Students' achievement and assessment issues

Some of the national presentations give partial answers to the first and second questions (what students really know and how are they assessed ?), but essentially show how these questions are difficult to tackle. In fact in most countries, hardly any empirical data is available concerning students' achievement at 16. Furthermore, when such data exist it is difficult to obtain them. To be clear, when data are available, there are aggregated data without the connected assessment items that would allow relevant comparisons, and when assessment items are available (for instance

examination questions) they are associated with passing rates and rarely with informative data concerning observed students' achievement.

The Third International Mathematics and Science Study has been unfortunately of few relevance for our project. First, only some of the concerned countries participated in TIMSS, second the TIMSS study has assessed 14 year old students (population 2), and 18 year old students (population 3) ; any inference made to 16 year students would be a bit hazardous.

The current OECD-PISA study might be more relevant : as a fact, PISA 2000 had a part devoted to mathematics ; not the main part but a part that should be informative enough. The fact that this study concerns 15 year old students in more than 30 countries including about all European countries makes this study of special interest for our project. Meanwhile, the first results will be known only by August 2001.

Therefore, investigating the achieved curriculum at 16, in Europe, is still a work to be done.

Concerning the fourth question (designing an assessment scheme), our group of experts quickly agreed to the idea that the question was premature. What has been considered more urgent, useful, and manageable , was to draw a picture as accurate as possible of the way 16 year old students are taught across our respective countries and what are the main problems teaching mathematics has to face. So that makes the bulk of the reports (the international one and the national presentations).

Meanwhile, the third question (what is worth to know ?) has always been present more or less explicitly in our discussions. Being a group of mathematicians and of mathematics educators, this even appeared to be quite the more relevant question.

Finally we decided to give a partial joint answer to the third and fourth question with proposing a bunch of mathematical tasks that can convey our feelings about what students should learn and how that learning might be assessed.

So we are proposing questions that we are considering of didactical value, worth to be used in assessment settings, but also that can be inserted in the flow of the teaching process. We did not try to deliver any ready-to-wear assessment items but more to suggest a possible style that can be adapted in each national context (as can do a *couturier*). Our ambition is to make these questions known and used across European countries and to get feed back from users (especially teachers) along with proposals for alternative and additional questions. We would like to consider these questions as a starting point for a European database of assessment questions.

# 3 - The use of the OECD PISA study

As our European mathematical education systems are on the way to be internationally assessed   by the OECD PISA 2000, 2003 and 2006 studies[1]/[2], we thought interesting to link our study to the

---

[1] OECD : 1999, Measuring Student Knowledge and - A new framework for assessment  (also available in French).

OECD : 1999, Measuring Student Knowledge and - The PISA 2000 Assessment of Reading, Mathematical and science literacy - Education and Skills  (also available in French).

[2] http://www.pisa.oecd.org/

PISA framework. On a one hand, the PISA framework seemed to be the more relevant for our purpose (and the one that takes more into account the current reflections and knowledge concerning the didactics of mathematics) ; on the other hand, we thought useful to help favouring a common language to address the students' knowledge and competencies issue across Europe.

The PISA study is aimed to assess mathematical literacy (along with language and sciences) of 15 year old students across countries.

Here is the PISA definition of mathematical literacy[3].

> **Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen.**

So PISA is deliberately aimed to assess at our "target 1" level *(for all,),* but at age 15. Meanwhile, the PISA framework is relevant for any age level and may be extended far beyond what is called here mathematical literacy.

# 4 - Reference questions

We name "reference question" any mathematical problem, task, teaching or assessment situation that the members of our group are considering :

- Relevant for exemplifying a class of situations of importance in mathematics (general relevance).
- Relevant for 1 or 2 dimensions of our 3 "targets" :
    - *Target 1 : For all*
    - *Target 2 : For those supposed they need mathematics for further studies*
    - *Target 3 : For those likely heading towards higher mathematics*
- Sufficiently important to justify subsequent pedagogical effort towards better student achievement on similar questions.
- Not too closely linked to a particular national curriculum or "didactical contract".

In this part of the study, we don't have much in mind any kind of assessment constraints. On the opposite, we aim to present a set of questions that we consider interesting for teaching and learning.

We don't focus on mathematical content but more on processes and on the way mathematical questions are understood and mathematical knowledge is likely to be activated in mathematical situations.

Part of those reference questions are *tasks* that in our view students by age 16 should be able to complete : either because they have already been exposed to situations of the like or because coping

---

[3] All texts borrowed to the OECD PISA documents are written in distinctive style (this one)

with them can be viewed as expected results of previous learning or still possibly because our group of experts think so, even if that doesn't fit current practices.

However, we aim also to present tasks in which 15 years old students should be able to show interest, to engage in, and achieve substantial part of the implied process. In our view, assessment can in no mean be only a question of success and failure ; students have also to be confronted with tasks in which full mastery is not expected (at least from all of them) but in which they might show their mathematical qualities in exploring, experimenting, modelling, linking mathematical situation with their mathematical formal knowledge,...). So, intermediate knowledge as well as mathematical attitudes should also be aimed in assessment settings.

Most of these questions might be entitled to be adapted to become part of tests and to contribute to build assessment scales and then, better inform reference proficiency levels. But that is not our first goal.

For the moment, we are more concerned by the mathematical education quality, and by consequential validity (S. Messick[4]) of further assessment tools that might be derived from our proposals than in the assessment process itself.

Here it is important to make clear that we want to encourage, in addition of traditional competencies and qualities pursued with mathematical education :

- Students' self confidence.
- Students' self assessment (to give importance to self evaluation and measure of confidence by the students themselves).
- Assurance in giving meaning to the acquired knowledge.
- Mathematical initiative and creativity.

We did not try to obtain any homogeneous set of questions. On the opposite, we want to stress on the need for diversity : diversity in learning settings as well as in assessment settings.

All the proposed reference questions are accompanied by an *Identity Card* featuring them.

To do that, we could have used the classification presented in our preliminary report :

| General Abilities | Mathematical World | Application of Math |
|---|---|---|
| Algorithms<br>Reasoning, deduction, proof<br>Language (using, creating, communicating ... symbols)<br>Visual thinking<br>Transfer<br>*Appreciation of mathematics, confidence to using it* | *Arithmetic*<br>*Variables, equations*<br>*Geometry*<br>*Data Analysis*<br>*Functions (incl. Graphs)* | Modelling<br>Investigation, (re)search<br>Approximate calculation<br>Computer(assisted learning.)<br>Self control (checking) |

---

[4] Messick, S.: 1995, *Values and Standards in Performance assessment : Issues, Findings, and Viewpoints,* Educational Measurement : Issues and Practice spécial issue, vol 14, n°4, NCME

As a fact, this typology was mainly aimed to address curriculum design and teaching issues. It is insufficient to describe mathematical tasks and, further on, to address the students' achievement and other assessment issues.

Therefore, as explained earlier, we are using a task analysis framework borrowed from the PISA 2003 project (only slightly modified) that addresses the following "dimensions" :

- ▪ *The conceptual fields or problem fields the question may principally refer to.*
- ▪ *The mathematical competencies mainly involved.*
- ▪ *The cognitive complexity of the task.*

# 5 : The PISA framework

## 5 - I : The "conceptual fields" or "problematic fields" the question may principally refer to

As in PISA, instead of focusing mainly on mathematical content, we prefer to put ahead the mathematical processes and the problem fields. That has been addressed in PISA as the "Big ideas" scheme. Note that the notion of "big ideas" matches well with our "dream questions" design. Both call for innovation and for going further deep towards meaningful mathematical learning.

**(Pisa quotation) The concept of "big ideas" is not new. In 1990, the Mathematical Sciences Education Board (Senechal, 1990) published "On the Shoulders of the Giant: New Approaches to Numeracy", which is a strong plea to help students delve deeper in order to find concepts that underlie mathematics and hence to reach a better understanding of their significance in the real world. For this it is necessary to explore ideas with deep roots in the mathematical sciences without concern for the limitations imposed by present school curricula.**

Other mathematicians support this idea, one of the better known publications being "Mathematics : The Science of Patterns"(Devlin, 1994, 1997).

The following list of mathematical big ideas is used in OECD/PISA to meet this requirement :

**P1** – **Quantity**

**P2** – **space and shape**

**P3** – **Change and relationship**

**P4** – **uncertainty**

Papers clarifying and exemplifying these "big ideas" have been written by the PISA mathematics expert group (cf. references) ; there is also comprehensive descriptions in the book quoted above (on the shoulder of the giants)[5].

---

[5] Steen, L. A (ed) : (1990), On the shoulders of the giants - New approaches to numeracy. National Academic Press (Washington)

Indeed, this classification may seem a bit too broad. As a fact it is a reduction of a larger classification that has been published elsewhere and even a reduction of a first one used by PISA 2000. Having in view to stay consistent with the PISA 2003 framework and as that did not hinder our views, we have preferred to stick to this part of the PISA framework.

Actually, many significant mathematical tasks can be referred to more than one conceptual field. As long as we want to give priority to content validity (and not to psychometrics validity), this is not an issue.

In addition with being referred to those conceptual fields, our questions refer also to the mathematical content they are directly or indirectly likely to address (see Questions Identity Card).

However, what we want help developing in students' learning and subsequently what we want our reference questions reflect is the large spectrum of competencies practising mathematics both require and help to develop. For that we could have used such or such framework already experimented in some of our countries (for instance the R. Gras classification often used in France), but we thought that a framework known and used in the international scene more appropriate. The more recent and the more in agreement with our views was, here again, the one developed by PISA.

## 5 - II : Mathematical competencies (still borrowed from PISA)

This [ ] is a non-hierarchical list of general mathematical skills that are relevant and pertinent to all levels of education. This list includes the following elements :

### C1. Mathematical thinking skill

This includes posing questions characteristic of mathematics ("Is there...?", "If so, how many?", "How do we find...?"); knowing the kinds of answers that mathematics offers to such questions; distinguishing between different kinds of statements (definitions, theorems, conjectures, hypotheses, examples, conditioned assertions); and understanding and handling the extent and limits of given mathematical concepts.

### C2. Mathematical argumentation skill

This includes knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning; following and assessing chains of mathematical arguments of different types; possessing a feel for heuristics ("What can(not) happen, and why?"); and creating mathematical arguments.

### C3. Modelling skill

This includes structuring the field or situation to be modelled; "mathematising" (translating "reality" into mathematical structures); "de-mathematising" (interpreting mathematical models in terms of "reality"); working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating about

the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.

## C4. Problem posing and solving skill

This includes posing, formulating, and defining different kinds of mathematical problems ("pure", "applied", "open-ended" and "closed"); and solving different kinds of mathematical problems in a variety of ways.

## C5. Representation skill

This includes decoding, interpreting and distinguishing between different forms of representation of mathematical objects and situations and the interrelationships between the various representations; choosing, and switching between, different forms of representation, according to situation and purpose.

## C6. Symbolic, formal and technical skill

This includes: decoding and interpreting symbolic and formal language and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; using variables, solving equations and undertaking calculations.

## C7. Communication skill

This includes expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others' written or oral statements about such matters.

## C8. Aids and tools skill

This includes knowing about, and being able to make use of, various aids and tools (including information technology tools) that may assist mathematical activity, and knowing about the limitations of such aids and tools.

## 5 - III : Cognitive complexity of the task (PISA competency classes)

We first intended to use the R. Gras taxonomy for describing and pointing out the level of task complexity, but for the moment we can stick to the classification proposed by PISA :

- **Class 1: reproduction, definitions, and computations.**
- **Class 2: connections and integration for problem solving.**
- **Class 3: mathematical thinking, generalisation and insight**

In a sense, this classification corresponds closely to the one formulated by R. Gras.

| PISA competency class | Class 1 | Class 2 | Class 3 |
|---|---|---|---|
| R. GRAS complexity level | A | B – C | D - E |

When we will need to refine particular reference question classification we could use the R. Gras extended taxonomy[6] (due to be soon translated into English).

Note that the cognitive taxonomy (as well as the competency classes) concerns task features (while not absolutely independent of the group of people the task is submitted to) and not the question difficulty. The difficulty relates always to a target group (even, but at a less extend, when using sophisticated methods as the "Item Response Theory"). The following PISA quotation doesn't contradict these assumptions.

[...PISA....] Each of the skills listed above is likely to play a role in all competency classes. That is, the skills do not belong within only one competency class. The classes form a conceptual continuum, from simple reproduction of facts and computational skills, to the competency of making connections between different strands in order to solve simple real-world problems, and to the third class, which involves the "mathematisation" (this term is discussed in detail below) of real-world problems and reflection on the solutions in the context of the problems, using mathematical thinking, reasoning and generalisation.

The above discussion suggests that the classes form a hierarchy, in the sense that a set of tasks requiring Class 3 competencies will in general be more difficult than a set of tasks requiring Class 2 competencies. However, this does not imply that Class 2 competencies are a prerequisite for each Class 3 competency. In fact, previous studies (de Lange, 1987; Shafer and Romberg, in press) show that it is not necessary to excel in Class 1 competencies to do well in Class 2 or 3, while students performing well in Class 3 may not necessarily excel in Class 1 competencies.

## Class 1 competencies: reproduction, definitions, computations

In this class, material typically featured in many standardised assessments and in comparative inter-national studies is dealt with. The class includes knowledge of facts, representation, recognition of equivalents, recalling of mathematical objects and properties, performance of routine procedures, application of standard algorithms and development of technical skills. Manipulation of expressions containing symbols and formulae in standard form, and calculations, are also competencies in this class. Items that assess competencies in this class can usually be in a multiple-choice or a restricted open-ended format.

This class relates in particular to the symbolic,formal and technical skill ....

## Class 2 competencies: connections and integration for problem solving

---

[6] Gras R. : 1977, *Contributions à l'étude expérimentale et à l'analyse de certaines acquisitions cognitives et de certains objectifs didactiques en mathématiques* - Thèse- université de RENNES.

In this class, connections between the different strands and domains in mathematics are of importance, and information must be integrated in order to solve simple problems. Students will therefore have to choose which strategies and mathematical tools to use. Although the problems are classified as non-routine, they require only a relatively low level of mathematisation.

In this class, students are also expected to handle different methods of representation, according to situation and purpose. The connections component also requires students to be able to distinguish and relate different statements such as definitions, claims, examples, conditioned assertions and proofs.

The class relates to several of the mathematical skills mentioned above. It is clear that solving the problems given in the example requires some reasoning or argumentation; hence it requires the use of mathematical argumentation skills. Further, the students need to "model" the problem in order to solve it –thus modelling skills are required. The problem solving itself requires problem-posing and problem-solving skills. When in the process of solving the problem the students use various forms of representation – a table, a chart or a drawing – this requires representation skills.

From the mathematical language point of view, decoding and interpreting symbolic and formal language and understanding its relationship to natural language is another important skill in this class. Items in this class are often placed within a context, and engage students in mathematical decision making.

## Class 3 competencies: mathematical thinking, generalisation and insight

For items in this class, students are asked to "mathematise" situations, that is, to recognise and extract the mathematics embedded in the situation and to use mathematics to solve the problem; to analyse; to interpret; to develop their own models and strategies and to present mathematical arguments, including proofs and generalisations.

These competencies include an analysis of the model and reflection on the process. In this class of competencies, students should not only be able to solve problems but also to pose problems.

All these competencies will function well only if students are able to communicate adequately in different ways: oral, written, visual, etc. Communication is regarded as a two-way process: students should be able to communicate their mathematical ideas as well as to understand the mathematical communications of others.

Finally, it is important to stress that students also need insight into the nature of mathematics, including cultural and historical elements, and the use of mathematics in other contexts and other curriculum areas that are amenable to mathematical modelling.

The competencies in this class often incorporate skills and competencies from other classes.

**"Mathematisation"**

In the PISA framework, mathematisation belongs to the class 3 of competency. We tried to point out what kind of mathematisation can be expected from our target 1 students' group and what kind of it can be expected from our Targets 2 and 3 students' groups.

Mathematisation, as it is used in OECD/PISA, refers to the organisation of perceived reality through the use of mathematical ideas and concepts. It is the organising activity according to which acquired knowledge and skills are used to discover unknown regularities, relationships and structures (Treffers and Goffree, 1985). This process is sometimes called horizontal mathematisation (Treffers, 1986). It requires activities such as:

- **identifying the specific mathematics in a general context**

- **schematising**

- **formulating and visualising a problem**

- **discovering relationships and regularities**

- **recognising similarities between different problems (de Lange, 1987)**

As soon as the problem has been transformed into a mathematical problem, it can be resolved with mathematical tools. That is, mathematical tools can be applied to manipulate and refine the mathematically modelled real-world problem. This process is referred to as vertical mathematisation and can be recognised in the following activities:

- **representing a relationship by means of a formula**

- **proving regularities**

- **refining and adjusting models**

- **combining and integrating models**

- **generalising**

Thus the process of mathematisation occurs in two different phases: horizontal mathematisation, which is the process of translating the real world into the mathematical world, and vertical mathematisation, that is, working on a problem within the mathematical world and using mathematical tools in order to solve the problem. Reflecting on the solution with respect to the original problem is an essential step in the process of mathematisation that seldom receives adequate attention.

One can argue that mathematisation occurs in all competency classes because, in any contextualised problem, one needs to identify the relevant mathematics. However, in OECD/PISA, the kind of mathematisation that is required in Competency Class 3 is of particular importance. It is that form of mathematisation that goes beyond the mere recognition of well-known problems.

# 6 - Reference Question Identity Card and Trial Records

The reference questions presented in the second part of this document (bundle of proposed reference questions) are accompanied by an Identity Card that tries to summarise information concerning especially :

- The origin of the question.

- Its classification according the PISA framework.

- Possible targets, according our 3 "target groups" :

    - For all

    - For those considered to need mathematics for further studies

    - For those likely to be heading towards higher mathematics.

- Its fitness to curriculum in some countries (likeliness a student aged 16 may have met similar questions : in lessons, assessment settings, or others activities related to school mathematics)

- Its expected achievement rates at 16 in some countries.

- Results of possible try out

- The kind of setting suggested for the question (class assignment, group work, assessment task, house or personal work…), specification for time allocation, material made available,…

| EMS REFERENCE QUESTION IDENTITY CARD | |
|---|---|
| NAME and Number of the Question : | **EMS XXX** |
| Origin of the question | |
| Problematic field («Big idea») | |
| Main contents supposed to be covered | |
| Competencies supposed to be implied | |
| Complexity class | |
| Target group | |
| Type of setting | |

| TRIAL COUNTRY | | |
|---|---|---|
| Fitness to curriculum | | |
| Expected present achievement rate at 16 | | |
| **Try out of the question** | *Context of the trial* | |
| | *Number of students* | |
| | *Results* | |